

The Logotropic Dark Fluid as a unification of dark matter and dark energy

Pierre-Henri Chavanis

Laboratoire de Physique Théorique, Université Paul Sabatier, 118 route de Narbonne 31062 Toulouse, France

We propose a heuristic unification of dark matter and dark energy in terms of a single “dark fluid” with a logotropic equation of state $P = A \ln(\rho/\rho_P)$, where ρ is the rest-mass density, $\rho_P = 5.16 \times 10^{99} \text{ g m}^{-3}$ is the Planck density, and A is the logotropic temperature. The energy density ϵ is the sum of a rest-mass energy term ρc^2 mimicking dark matter and an internal energy term $u(\rho) = -P(\rho) - A$ mimicking dark energy. The logotropic temperature is approximately given by $A \simeq \rho_\Lambda c^2 / \ln(\rho_P/\rho_\Lambda) \simeq \rho_\Lambda c^2 / [123 \ln(10)]$, where $\rho_\Lambda = 6.72 \times 10^{-24} \text{ g m}^{-3}$ is the cosmological density and 123 is the famous number appearing in the ratio $\rho_P/\rho_\Lambda \sim 10^{123}$ between the Planck density and the cosmological density. More precisely, we obtain $A = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}$ that we interpret as a fundamental constant. At the cosmological scale, this model fullfills the same observational constraints as the Λ CDM model (they will differ in about 25 Gyrs when the logotropic universe becomes phantom). However, the logotropic dark fluid has a nonzero speed of sound and a nonzero Jeans length which, at the beginning of the matter era, is about $\lambda_J = 40.4 \text{ pc}$, in agreement with the minimum size of the dark matter halos observed in the universe. At the galactic scale, the logotropic pressure balances gravitational attraction and solves the cusp problem and the missing satellite problem. The logotropic equation of state generates a universal rotation curve that agrees with the empirical Burkert profile of dark matter halos up to the halo radius. In addition, it implies that all the dark matter halos have the same surface density $\Sigma_0 = \rho_0 r_h = 141 M_\odot/\text{pc}^2$ and that the mass of dwarf galaxies enclosed within a sphere of fixed radius $r_u = 300 \text{ pc}$ has the same value $M_{300} = 1.93 \times 10^7 M_\odot$, in remarkable agreement with the observations. It also implies the Tully-Fisher relation $M_b/v_h^4 = 44 M_\odot \text{ km}^{-4} \text{ s}^4$. We stress that there is no free parameter in our model (we predict the values of Σ_0 , M_{300} and M_b/v_h^4 in terms of fundamental constants). We sketch a justification of the logotropic equation of state in relation to the Cardassian model (motivated by the existence of extra-dimensions) and in relation to Tsallis generalized thermodynamics. We also develop a scalar field theory based on a Gross-Pitaevskii equation with an inverted quadratic potential, or on a Klein-Gordon equation with a logarithmic potential.

PACS numbers: 95.30.Sf, 95.35.+d, 95.36.+x, 98.62.Gq, 98.80.-k

I. INTRODUCTION

The nature of dark matter (DM) and dark energy (DE) is still unknown and remains one of the greatest mysteries of modern cosmology. DM has been introduced in astrophysics to account for the missing mass of the galaxies inferred from the virial theorem [1] and to explain their flat rotation curves [2]. DE has been introduced in cosmology to account for the present acceleration of the expansion of the universe [3]. In the standard cold dark matter (Λ CDM) model, DM is represented by a pressureless fluid and DE is ascribed to the cosmological constant Λ introduced by Einstein [4]. The Λ CDM model works remarkably well at the cosmological scale but it encounters serious problems at the galactic scale. In particular, it predicts that DM halos should be cuspy [5] while observations reveal that they have a flat core [6]. On the other hand, the Λ CDM model predicts an overabundance of small-scale structures (subhalos/satellites), much more than what is observed around the Milky Way [7]. These problems are referred to as the “cusp problem” and “missing satellite problem”. The expression “small-scale crisis of CDM” has been coined.

There are also unexplained important observational results. For example, it is an empirical fact that the surface density of galaxies has the same value $\Sigma_0 = \rho_0 r_h = 141_{-52}^{+83} M_\odot/\text{pc}^2$ even if their sizes and masses vary by sev-

eral orders of magnitude (up to 14 orders of magnitude in luminosity) [8]. On the other hand, it is known that the asymptotic circular velocity of the galaxies is related to their baryonic mass by the Tully-Fisher (TF) relation $M_b/v_h^4 = 47 \pm 6 M_\odot \text{ km}^{-4} \text{ s}^4$ [9, 10]. Finally, Strigari *et al.* [11] have shown that all dwarf spheroidal galaxies (dSphs) of the Milky Way have the same total DM mass contained within a radius of $r_u = 300 \text{ pc}$. From the observations, they obtained $\log(M_{300}/M_\odot) = 7.0_{-0.4}^{+0.3}$. To our knowledge, there is no theoretical explanation of these observational results.

The small scale problems of the Λ CDM model are related to the assumption that DM is pressureless. This assumption is valid if DM is made of weakly interacting massive particles (WIMPs) with a mass in the GeV-TeV range. These particles freeze out from thermal equilibrium in the early universe and, as a consequence of this decoupling, cool off rapidly as the universe expands. In order to solve the small-scale crisis of CDM, some authors have developed alternative models of DM. For example, it has been proposed that DM halos are made of fermions (such as sterile neutrinos) with a mass in the keV range [12, 13], or bosons (such as axions) in the form of Bose-Einstein condensates (BECs) with a mass ranging from 10^{-2} eV to 10^{-20} eV depending whether the bosons interact or not [14]. In these models, the quantum pressure prevents gravitational collapse and leads to cores instead

of cusps. These models sometimes provide a good fit of the rotation curves of galaxies but they do not explain the universality (and the values) of Σ_0 , M_b/v_h^4 , and M_{300} .

On the other hand, at the cosmological scale, despite its success at explaining many observations, the Λ CDM model has to face two theoretical problems. The first one is the cosmic coincidence problem, namely why the ratio of DE and DM is of order unity today if they are two different entities [15]. The second one is the cosmological constant problem [16]. The cosmological constant Λ is equivalent to a constant energy density $\epsilon_\Lambda = \rho_\Lambda c^2 = \Lambda c^2/8\pi G$ associated with an equation of state $P = -\epsilon$ involving a negative pressure. Some authors [17] have proposed to interpret the cosmological constant in terms of the vacuum energy. Cosmological observations lead to the value $\rho_\Lambda = \Lambda/8\pi G = 6.72 \times 10^{-24} \text{ g m}^{-3}$ of the cosmological density (DE). However, particle physics and quantum field theory predict that the vacuum energy should be of the order of the Planck density $\rho_P = c^5/\hbar G^2 = 5.16 \times 10^{99} \text{ g m}^{-3}$. The ratio between the Planck density ρ_P and the cosmological density ρ_Λ is

$$\frac{\rho_P}{\rho_\Lambda} \sim 10^{123}, \quad (1)$$

so these quantities differ by 123 orders of magnitude! This is the origin of the cosmological constant problem. To circumvent this problem, some authors have proposed to abandon the cosmological constant Λ and to explain the acceleration of the universe in terms of a dark energy with a time-varying density associated with a scalar field called “quintessence” [18]. As an alternative to quintessence, Kamenshchik *et al.* [19] have proposed a heuristic unification of DM and DE in terms of an exotic fluid with an equation of state $P = -A/\epsilon$ called the Chaplygin gas. This equation of state provides a model of universe that behaves as a pressureless fluid (DM) at early times, and as a fluid with a constant energy density (DE) at late times, yielding an exponential acceleration similar to the effect of the cosmological constant. However, in the intermediate regime of interest, this model does not give a good agreement with the observations [20] so that various generalizations of the Chaplygin gas model have been considered. In this Letter, we propose a new model based on a logotropic equation of state [21] that seems to give a solution to all the problems mentioned above and, most importantly, that *predicts* the correct values of Σ_0 , M_b/v_h^4 , and M_{300} with remarkable accuracy, and without free parameter.

II. LOGOTROPIC COSMOLOGY

A. The logotropic dark fluid

The Friedmann equations for a flat universe without cosmological constant are [22]:

$$\frac{d\epsilon}{dt} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0, \quad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon, \quad (2)$$

where $\epsilon(t)$ is the energy density, $P(t)$ is the pressure, $a(t)$ is the scale factor, and $H = \dot{a}/a$ is the Hubble parameter.

For a relativistic fluid at $T = 0$, or for an adiabatic evolution (which is the case for a perfect fluid), the first law of thermodynamics reduces to [22]:

$$d\epsilon = \frac{P + \epsilon}{\rho} d\rho, \quad (3)$$

where ρ is the rest-mass density. Combined with the equation of continuity (2), we get

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho = 0 \Rightarrow \rho = \frac{\rho_0}{a^3}, \quad (4)$$

where ρ_0 is the present value of the rest-mass density, and the present value of the scale factor is taken to be $a_0 = 1$. This equation, which expresses the conservation of the rest-mass, is valid for an arbitrary equation of state.

For an equation of state specified under the form $P = P(\rho)$, Eq. (3) can be integrated to obtain the relation between the energy density ϵ and the rest-mass density. We obtain

$$\epsilon = \rho c^2 + \rho \int^\rho \frac{P(\rho')}{\rho'^2} d\rho' = \rho c^2 + u(\rho), \quad (5)$$

where the constant of integration is set equal to zero. We note that $u(\rho)$ can be interpreted as an internal energy density. Therefore, the energy density ϵ is the sum of the rest-mass energy ρc^2 and the internal energy $u(\rho)$. The rest-mass energy is positive while the internal energy can be positive or negative. Of course, the total energy $\epsilon = \rho c^2 + u(\rho)$ is always positive.

We assume that the universe is filled with a single dark fluid described by a logotropic equation of state

$$P = A \ln \left(\frac{\rho}{\rho_P} \right), \quad (6)$$

where A is the logotropic temperature (determined below) and $\rho_P = 5.16 \times 10^{99} \text{ g m}^{-3}$ is the Planck density. It will be called the Logotropic Dark Fluid (LDF). Using Eqs. (5) and (6), the relation between the energy density and the rest-mass density is

$$\epsilon = \rho c^2 - A \ln \left(\frac{\rho}{\rho_P} \right) - A = \rho c^2 + u(\rho). \quad (7)$$

The energy density is the sum of two terms: a rest-mass energy term $\rho c^2 \propto a^{-3}$ that mimics DM and an internal energy term $u(\rho) = -P(\rho) - A$ that mimics DE. This decomposition leads to a natural, and physical, unification of DM and DE and elucidates their mysterious nature. We note that the pressure is related to the internal energy by $P = -u - A$. Combining Eqs. (6) and (7), we obtain $\epsilon = \rho_P c^2 e^{P/A} - P - A$ which determines, by inversion, the equation of state $P(\epsilon)$. From Eqs. (4), (6) and (7), we get $P = A \ln(\rho_0/\rho_P a^3)$ and $\epsilon = \rho_0 c^2/a^3 - A \ln(\rho_0/\rho_P a^3) - A$. We note that the internal energy $u = -A \ln(\rho/\rho_P) - A$ is positive for $\rho < \rho_P/e$ and negative for $\rho > \rho_P/e$.

In the early universe ($a \rightarrow 0$, $\rho \rightarrow +\infty$), the rest-mass energy (DM) dominates, so that

$$\epsilon \sim \rho c^2 \sim \frac{\rho_0 c^2}{a^3}, \quad P \sim A \ln \left(\frac{\epsilon}{\rho_P c^2} \right). \quad (8)$$

For small values of the scale factor, we recover the results of the CDM model ($P = 0$) since $\epsilon \propto a^{-3}$. In the late universe ($a \rightarrow +\infty$, $\rho \rightarrow 0$), the internal energy (DE) dominates, and we have

$$\epsilon \sim -A \ln \left(\frac{\rho}{\rho_P} \right) \sim 3A \ln a, \quad P \sim -\epsilon. \quad (9)$$

We note that the equation of state $P(\epsilon)$ behaves asymptotically as $P \sim -\epsilon$, similarly to the usual equation of state of DE. It is interesting to recover the equation of state $P = -\epsilon$ from the logotropic model (6). This was not obvious *a priori*.

B. The logotropic temperature

Since, in our model, the rest-mass energy of the dark fluid mimics DM, we identify ρ_0 with the present density of DM. We thus set $\rho_0 = \Omega_{m,0} \epsilon_0 / c^2 = 2.54 \times 10^{-24} \text{ g m}^{-3}$, where $\epsilon_0 / c^2 = 3H_0^2 / 8\pi G = 9.26 \times 10^{-24} \text{ g m}^{-3}$ is the present energy density of the universe (we have taken $H_0 = 70.2 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.275 \times 10^{-18} \text{ s}^{-1}$) and $\Omega_{m,0} = 0.274$ is the present fraction of DM (we also include baryonic matter). As a result, the present internal energy of the dark fluid $u_0 / c^2 = \epsilon_0 / c^2 - \rho_0$ is identified with the present density of DE $\rho_\Lambda = (1 - \Omega_{m,0}) \epsilon_0 / c^2 = 6.72 \times 10^{-24} \text{ g m}^{-3}$ where $\Omega_{\Lambda,0} = 1 - \Omega_{m,0} = 0.726$ is the present fraction of DE.

Applying Eq. (7) at $a = 1$, we obtain the identity

$$\frac{\rho_P}{\rho_\Lambda} = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} e^{1+1/B}, \quad (10)$$

where we have defined the dimensionless logotropic temperature B through the relation $A = B \rho_\Lambda c^2$. This identity is strikingly similar to Eq. (1) which appears in relation to the cosmological constant problem. In the present context, the identity (10) determines the logotropic temperature B . Qualitatively, $B \simeq 1 / \ln(\rho_P / \rho_\Lambda) \simeq 1 / [123 \ln(10)]$. This gives a new interpretation to the famous number $123 \simeq \log(\rho_P / \rho_\Lambda)$ as being the inverse logotropic temperature. More precisely, we obtain

$$B = \frac{1}{\ln \left(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}} \frac{\rho_P}{\rho_\Lambda} \right) - 1} = 3.53 \times 10^{-3} \quad (11)$$

and

$$A = B \rho_\Lambda c^2 = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}. \quad (12)$$

As a result, there is no free parameter in our model. The logotropic temperature is determined from the Planck

density ρ_P and the cosmological density ρ_Λ (itself obtained from the Hubble constant H_0 and the fraction of DM $\Omega_{m,0}$). From now on, we shall regard A as a fundamental constant that supersedes the cosmological constant. We note that it depends on all the fundamental constants of physics \hbar , G , c , and Λ [see Eqs. (11) and (12)].

After simple manipulations, the rest-mass density, the pressure and the energy density of the LDF can be expressed in terms of B as

$$\frac{\rho c^2}{\epsilon_0} = \frac{\Omega_{m,0}}{a^3}, \quad \frac{P}{\epsilon_\Lambda} = -B - 1 + B \ln \left(\frac{\rho c^2}{\epsilon_0 \Omega_{m,0}} \right), \quad (13)$$

$$\frac{P}{\epsilon_\Lambda} = -B - 1 - 3B \ln a, \quad (14)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\rho c^2}{\epsilon_0} + (1 - \Omega_{m,0}) \left[1 + B \ln \left(\frac{\Omega_{m,0} \epsilon_0}{\rho c^2} \right) \right], \quad (15)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})(1 + 3B \ln a), \quad (16)$$

$$\frac{\epsilon}{\epsilon_0} = \Omega_{m,0} e^{(B+1)/B} e^{P/B\epsilon_\Lambda} - (1 - \Omega_{m,0}) \left(\frac{P}{\epsilon_\Lambda} + B \right). \quad (17)$$

The Λ CDM model is recovered for $B = 0$, i.e., $\epsilon / \epsilon_0 = \rho c^2 / \epsilon_0 + (1 - \Omega_{m,0})$, $\epsilon / \epsilon_0 = \Omega_{m,0} / a^3 + 1 - \Omega_{m,0}$, and $P = -\epsilon_\Lambda$. The Λ CDM model is equivalent to a constant negative pressure $P = -\epsilon_\Lambda$ [23] and to the relation $\epsilon = \rho c^2 + \epsilon_\Lambda$ between the energy density and the rest-mass density. According to Eq. (10), the condition $B = 0$ in the logotropic model corresponds to $\rho_P = +\infty$, hence $\hbar = 0$. Therefore, the fact that B is small but nonzero as vindicated by the observations (see below) shows that quantum mechanics ($\hbar \neq 0$) plays a role in the late universe in relation to DE.

C. Evolution of the logotropic universe

The relation between the energy density and the rest-mass density [see Eq. (15)] is plotted in Fig. 1. The evolution of the energy density with the scale factor [see Eq. (16)] is plotted in Fig. 2. The universe starts at $a = 0$ with an infinite rest-mass density ($\rho \rightarrow +\infty$) and an infinite energy density ($\epsilon \rightarrow +\infty$).¹ The rest-mass density decreases as a increases. The energy density first

¹ Of course, our model that attempts to unify DM and DE is only valid at sufficiently late times, typically for $a > a_i = 10^{-4}$, after the inflation and the radiation eras. Therefore, the limit $a \rightarrow 0$ is here formal.

decreases as a increases (i.e. ρ decreases), reaches a minimum $\epsilon_M = -A \ln(A/\rho_P c^2)$ at $a_M = (\rho_0 c^2/A)^{1/3}$ (i.e. $\rho_M = A/c^2$), then increases as a increases (i.e. ρ decreases) further, and tends to $\epsilon \rightarrow +\infty$ as $a \rightarrow +\infty$ (i.e. $\rho \rightarrow 0$). The branch $a \leq a_M$ (i.e. $\rho \geq \rho_M$) corresponds to a normal behavior in which the energy density decreases as the scale factor increases. The branch $a \geq a_M$ (i.e. $\rho \leq \rho_M$) corresponds to a phantom behavior [24] in which the energy density increases as the scale factor increases. We note that A is equal to the rest-mass energy at the point where the universe becomes phantom.

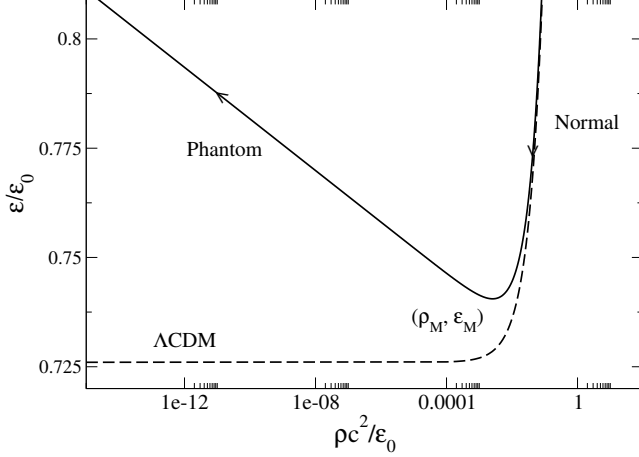


FIG. 1: Relation between the energy density ϵ and the rest-mass density ρ in the logotropic model. It is compared with the relation $\epsilon = \rho c^2 + \epsilon_\Lambda$ corresponding to the Λ CDM model. The energy density presents a minimum $(\epsilon/\epsilon_0)_M = 0.7405$ at $\rho_M c^2/\epsilon_0 = 2.56 \times 10^{-3}$ separating the normal universe and the phantom universe.

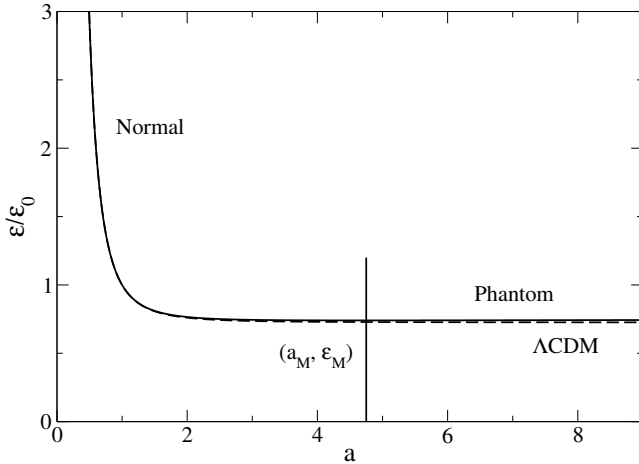


FIG. 2: Evolution of the energy density as a function of the scale factor in the logotropic model. It is compared with the Λ CDM model. The energy density presents a minimum $(\epsilon/\epsilon_0)_M = 0.7405$ at $a_M = 4.75$.

The evolution of the pressure with the scale factor [see Eq. (14)] is plotted in Fig. 3. The pressure decreases as

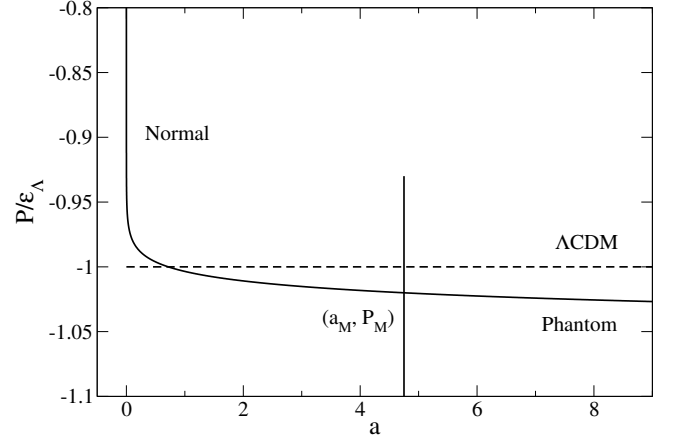


FIG. 3: Evolution of the pressure as a function of the scale factor in the logotropic model. It is compared with the Λ CDM model where $P = -\epsilon_\Lambda$. The pressure becomes negative at $a_w = 7.00 \times 10^{-42}$. The point separating the normal universe from the phantom universe is located at $a_M = 4.75$ and $P_M/\epsilon_\Lambda = -1.02$.

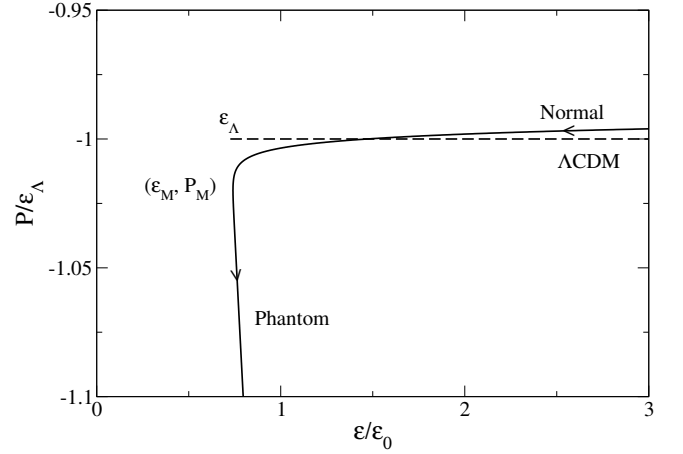


FIG. 4: Equation of state $P(\epsilon)$ of the logotropic model. It is compared with the equation of state $P = -\epsilon_\Lambda$ of the Λ CDM model.

a increases (i.e. ρ decreases). It starts from $P \rightarrow +\infty$ at $a = 0$ (i.e. $\rho \rightarrow +\infty$, $\epsilon \rightarrow +\infty$), vanishes at $a_w = (\rho_0/\rho_P)^{1/3}$, achieves the value $P_M = -\epsilon_M$ at a_M (i.e. ρ_M, ϵ_M) and tends to $P \rightarrow -\infty$ as $a \rightarrow +\infty$ (i.e. $\rho \rightarrow 0$, $\epsilon \rightarrow +\infty$). The equation of state $P(\epsilon)$ [see Eq. (17)] is defined for $\epsilon \geq \epsilon_M$ and has two branches corresponding to a normal universe ($P \geq P_M$) and a phantom universe ($P \leq P_M$), as shown in Fig. 4. Therefore, the equation of state $P(\epsilon)$ is multi-valued.

The speed of sound c_s , defined by $c_s^2 = P'(\epsilon)c^2$, is given by $c_s^2/c^2 = 1/(\rho c^2/A - 1) = 1/[(a_M/a)^3 - 1]$. It is real for $a < a_M$ (i.e. when the universe is normal) and imaginary for $a > a_M$ (i.e. when the universe is phantom). The relation between the speed of sound and the scale factor is plotted in Fig. 5.

Solving the Friedmann equation (2) with the energy

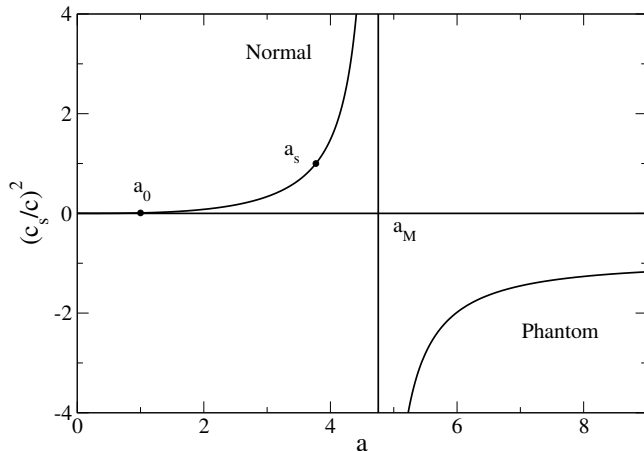


FIG. 5: Evolution of the speed of sound with the scale factor in the logotropic model. For the Λ CDM model, $c_s = 0$. The speed of sound is equal to the speed of light ($c_s = c$) at $a_S = a_M/2^{1/3} = 3.77$. At the present time ($a = a_0 = 1$), we have $(c_s/c)^2 = 1/(a_M^3 - 1) = 9.42 \times 10^{-3}$.

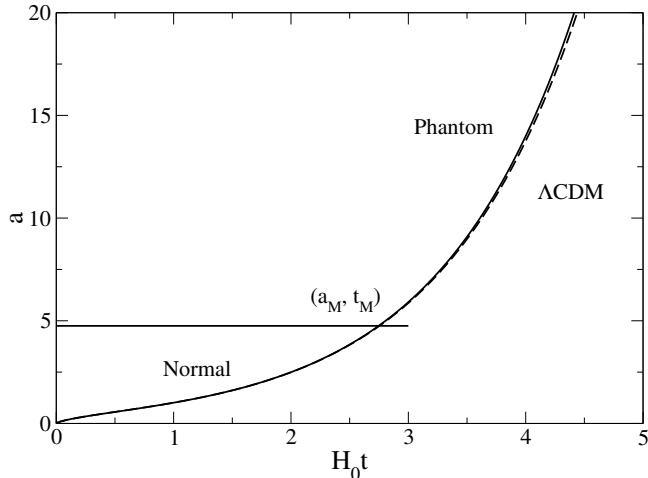


FIG. 6: Temporal evolution of the scale factor in the logotropic model as compared to the Λ CDM model. The age of the universe is $t_0 = 13.8$ Gyrs. In the logotropic model, the universe accelerates at $t_c = 7.19$ Gyrs, the speed of sound exceeds the speed of light at $t_s = 34.5$ Gyrs, and the universe becomes phantom at $t_M = 38.3$ Gyrs (at present $w_0 = P_0/\epsilon_0 = -0.729$).

density (16), we obtain the evolution of the scale factor represented in Fig. 6. For $t \rightarrow 0$, we have $a \sim (3\sqrt{\Omega_{m,0}}H_0 t/2)^{2/3}$ and $\epsilon/\epsilon_0 \sim 4/(9H_0^2 t^2)$ as in the Einstein-de Sitter (EdS) model. For $t \rightarrow +\infty$, we have $a \propto e^{3B(1-\Omega_{m,0})H_0^2 t^2/4}$ and $\epsilon/\epsilon_0 \sim [3B(1-\Omega_{m,0})H_0 t/2]^2$. This asymptotic solution, which is valid in the regime where the universe is phantom, has a super-de Sitter behavior. There is no future finite time singularity (no Big Rip). The scale factor, the energy density and the pressure become infinite in infinite time (Little Rip).

D. Cosmological implications

From the observational viewpoint, there is no visible difference between the logotropic model and the Λ CDM model at large scales. Differences will appear in about 25 Gyrs, when the universe becomes phantom (this aspect will be developed in a future work). However, this moment is very remote in the future, and for the time being, the logotropic model and the Λ CDM model behave similarly (see Figs. 2 and 6). This is satisfactory since the Λ CDM model works well at the cosmological scale. On a theoretical point of view, the logotropic model has several advantages with respect to the Λ CDM model. In our model, DM and DE are the manifestation of a single dark fluid described by a unique equation of state. Therefore, there is no cosmic coincidence problem. On the other hand, the cosmological constant problem of Eq. (1) is translated into an equation (10) that determines the logotropic temperature $A \simeq \epsilon_\Lambda/[123 \ln(10)]$.

An important difference between the Λ CDM model and the logotropic model concerns the speed of sound c_s defined by $c_s^2/c^2 = P'(\epsilon)$. In the Λ CDM model, since $P = 0$ (actually, $P = -\epsilon_\Lambda$), the speed of sound $c_s = 0$. As a result, the Jeans length is zero ($\lambda_J = 0$), implying that the homogeneous background is unstable at all scales so that halos of any size should be observed in principle. However, this is not the case. There does not seem to be halos with a size smaller than $R_{\min} \sim 10$ pc. Contrary to the Λ CDM model, the logotropic model has a nonzero speed of sound, hence a nonzero Jeans length. We can obtain an estimate of the Jeans length λ_J at the beginning of the matter era where perturbations start to grow. We assume that the matter era starts at $a_i = 10^{-4}$, corresponding to the epoch of matter-radiation equality. In this era, we can make the approximation $\epsilon = \rho c^2$, so the Jeans wavenumber is given by $k_J^2 = 4\pi G \rho a^2 / c_s^2$ [22], where $c_s^2 = P'(\rho) = A/\rho$. At $a_i = 10^{-4}$, we find $\rho_i = 2.54 \times 10^{-12} \text{g/m}^3$ and $(c_s^2/c^2)_i = 9.33 \times 10^{-15}$. This leads to a Jeans length $\lambda_J = 2\pi/k_J = 1.25 \times 10^{18} \text{m} = 40.4 \text{pc}$ which is of the order of magnitude of the smallest known dark matter halos such as Willman I ($r_h = 33 \text{pc}$) (see Table 2 of [25]). We predict that there should not exist halos of smaller size since the perturbations are stable for $\lambda < \lambda_J$. This is in agreement with the observations, unlike the Λ CDM model. Therefore, a small but finite value of B , yielding a nonzero speed of sound and a nonzero Jeans length, is able to account for the minimum observed size of dark matter halos in the universe. It also puts a cut-off in the density power spectrum of the perturbations and sharply suppresses small-scale linear power. This may be a way to solve the missing satellite problem.

III. LOGOTROPIC DARK MATTER HALOS

The interest of the logotropic model becomes evident when it is applied to DM halos. We assume that

DM halos are described by the logotropic equation of state (6) with the logotropic temperature $A = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}$ determined previously, viewed as a fundamental constant. At the galactic scale, we can use Newtonian gravity. Combining the condition of hydrostatic equilibrium $\nabla P + \rho \nabla \Phi = \mathbf{0}$ with the Poisson equation $\Delta \Phi = 4\pi G \rho$, assuming spherical symmetry, and introducing the notations $\theta = \rho_0/\rho$ and $\xi = r/r_0$, where ρ_0 is the central density and

$$r_0 = \left(\frac{A}{4\pi G \rho_0^2} \right)^{1/2} \quad (18)$$

is the logotropic core radius, we obtain the differential equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = \frac{1}{\theta} \quad (19)$$

with $\theta(0) = 1$ and $\theta'(0) = 0$. It can be viewed as a Lane-Emden equation of index $n = -1$ [26]. This equation has a simple analytical solution $\rho_s = (A/8\pi G)^{1/2} r^{-1}$ called the singular logotropic sphere because it diverges at the origin [21].² The regular solutions must be computed numerically. They have a flat core and behave as $\rho \sim (A/8\pi G)^{1/2} r^{-1}$ for $r \rightarrow +\infty$. Since the logotropic spheres are homologous, they generate a *universal* DM profile. Indeed, if we rescale the density by the central density ρ_0 and the radial distance by the core radius r_0 , we get an invariant density profile $1/\theta(\xi)$. We note that the total mass of a logotropic sphere is infinite because of the slow decay of the density. This means that the logotropic distribution cannot describe the whole cluster. It is valid only in the core. At larger distances, we must take into account complex physical processes such as tidal effects and incomplete relaxation that steepen the density profile (see, e.g., [13]). It is usually found that the density profiles of DM halos decrease at large distances as r^{-3} [5, 6]. Since we do not take these complicated processes into account, our logotropic model is only valid up to a few values of the core radius r_0 . However, this is sufficient to determine the physical characteristics of DM halos.

Using the Lane-Emden equation (19) the mass profile $M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$ is given by $M(r) = 4\pi \rho_0 r_0^3 \xi^2 \theta'(\xi)$. The circular velocity defined by $v_c^2(r) = GM(r)/r$ can be expressed as $v_c^2(r) = 4\pi G \rho_0 r_0^2 \xi \theta'(\xi)$. We define the halo radius r_h as the radius at which $\rho/\rho_0 = 1/4$. The dimensionless halo radius is the solution of the equation $\theta(\xi_h) = 4$. We numerically find $\xi_h = 5.8458$ and $\theta'(\xi_h) = 0.69343$. Then, $r_h = \xi_h r_0$. The normalized halo mass at the halo radius is given by

$$\frac{M_h}{\rho_0 r_h^3} = 4\pi \frac{\theta'(\xi_h)}{\xi_h} = 1.49. \quad (20)$$

This value is relatively close to the value $M_h/\rho_0 r_h^3 = 1.60$ [12, 13] obtained from the empirical Burkert profile [6] that provides a good fit of DM halos. On the other hand, the universal rotation curve predicted by the logotropic model is very close to the Burkert profile up to the halo radius, i.e. for $r \leq r_h$ (see Fig. 7). Very recently, Burkert [27] observed that the density profile of real DM halos behaves approximately as r^{-1} close to the halo radius. Interestingly, we note that the exponent -1 precisely corresponds to the characteristic exponent of the logotropes.

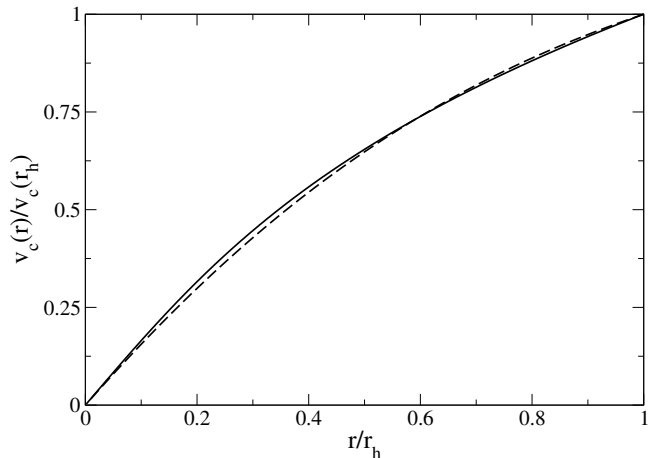


FIG. 7: Circular velocity profile of a logotropic sphere (solid line) in the region $r \leq r_h$ where the model is valid. It is compared with the Burkert profile (dashed line).

In addition to these already encouraging results, the logotropic equation of state has a very interesting property. According to Eq. (18), the surface density of the logotropic sphere is given by

$$\Sigma_0 \equiv \rho_0 r_h = \left(\frac{A}{4\pi G} \right)^{1/2} \xi_h. \quad (21)$$

Since the logotropic temperature A is the same for all the halos (a consequence of our approach where we view A as a fundamental constant), this implies that the surface density of the DM halos should be the same. This is precisely what is observed [8]. Using the value of the logotropic temperature given by Eq. (12), we get $\Sigma_0^{\text{th}} = 141 M_\odot/\text{pc}^2$ which coincides with the best-fit value $\Sigma_0^{\text{obs}} = 141_{-52}^{+83} M_\odot/\text{pc}^2$ of the surface density of DM halos [8]. This agreement is remarkable since there is no free parameter in our model. Furthermore, it is non trivial since the constant A depends, through Eqs. (11) and (12), on the Planck density ρ_P and on the cosmological density ρ_Λ . This suggests that there is something deep behind these relations.

There are interesting consequences of this result. According to Eq. (20), the mass of the halos calculated at the halo radius r_h is given by $M_h = 1.49 \Sigma_0 r_h^2$. On the other hand, the circular velocity at the halo radius is $v_h^2 = GM_h/r_h = 1.49 \Sigma_0 G r_h$. Since the surface

² We note, parenthetically, that this singular solution $\propto r^{-1}$ is similar to NFW cusps [5].

density of the dark matter halos is constant, we obtain $M_h/M_\odot = 210(r_h/\text{pc})^2 \propto r_h^2$ and $(v_h/\text{km s}^{-1})^2 = 0.905(r_h/\text{pc}) \propto r_h$. These scalings are consistent with the observations [12]. Furthermore, introducing the baryon mass $M_b = f_b M_h$ where $f_b \sim 0.17$ is the cosmic baryon fraction [10], we get

$$\frac{M_b}{v_h^4} = \frac{f_b}{1.49\Sigma_0 G^2} = \frac{f_b}{\theta'(\xi_h)(4\pi A G^3)^{1/2}}. \quad (22)$$

Therefore, $v_h^4 \propto M_h$ which is the TF relation. More precisely, we predict $(M_b/v_h^4)^{\text{th}} = 44 M_\odot \text{km}^{-4} \text{s}^4$ which is close to the observed value $(M_b/v_h^4)^{\text{obs}} = 47 \pm 6 M_\odot \text{km}^{-4} \text{s}^4$ [10] (we obtain a perfect agreement by taking $f_b = 0.18$).³

The logotropic equation of state also explains the observation of Strigari *et al.* [11] that all dSphs of the Milky Way have the same total dark matter mass M_{300} contained within a radius $r_u = 300 \text{ pc}$, namely $\log(M_{300}^{\text{obs}}/M_\odot) = 7.0_{-0.4}^{+0.3}$. Using $M_{300} = 4\pi\rho_0 r_0^3 \xi_u^2 \theta'(\xi_u)$, $\xi_u = r_u/r_0$ and $r_0 = r_h/\xi_h$, we obtain $M_{300} = 4\pi\rho_0 r_h (r_u^2/\xi_h) \theta'(\xi_h r_u/r_h)$. The logotropic distribution has the asymptotic behavior $\theta(\xi) \sim \xi/\sqrt{2}$ for $\xi \rightarrow +\infty$ [21]. For the dSphs considered in [11], $\xi_h r_u/r_h \gg 1$ (see Table 2 of [25]) so $\theta'(\xi_h r_u/r_h)$ can be replaced by its asymptotic value $1/\sqrt{2}$. This yields

$$M_{300} = \frac{4\pi\Sigma_0 r_u^2}{\xi_h \sqrt{2}} = r_u^2 \left(\frac{2\pi A}{G} \right)^{1/2}, \quad (23)$$

which is a constant, in agreement with the claim of Strigari *et al.* [11]. We note that the constancy of M_{300} is due to the universality of A . Furthermore, the numerical application gives $M_{300}^{\text{th}} = 1.93 \times 10^7 M_\odot$, leading to $\log(M_{300}^{\text{th}}/M_\odot) = 7.28$ in very good agreement with the observational value.

In conclusion, the logotropic equation of state can simultaneously account for cosmological constraints (with the same level of precision as the Λ CDM model) and explain properties of DM halos (their minimum size R_{min} , their surface density Σ_0 , their mass M_{300} , and the TF ratio M_b/v_h^4) that were not explained so far. This may be a hint that DM and DE are the manifestations of a single dark fluid. The best illustration of this “unification” is that we have obtained the value of A from cosmological constraints [see Eqs. (11) and (12)], and that this value accounts for the universality of the surface density Σ_0 and

mass M_{300} of DM halos [see Eqs. (21) and (23)], as well as for the TF relation [see Eq. (22)]. Assuming that this agreement is not a coincidence (the perfect agreement between the predicted values of Σ_0 , M_{300} , M_b/v_h^4 and the observations is a strong support to our approach), the next step is to justify the logotropic equation of state. We sketch below several possible justifications based on extra-dimensions (Cardassian model), generalized thermodynamics, and field theory.

IV. POSSIBLE JUSTIFICATIONS OF THE LOGOTROPIC EQUATION OF STATE

A. Cardassian model

Freese and Lewis [29], in their so-called Cardassian model, have proposed to explain the accelerated expansion of the universe in terms of a modified Friedmann equation of the form

$$H^2 = \frac{8\pi G}{3}\rho + \nu(\rho), \quad (24)$$

where $\rho = \rho_0 a^{-3}$ is the rest-mass density and $\nu(\rho)$ is a “new” term which characterizes the model. In the early universe, the term $\nu(\rho)$ is negligible and one recovers the usual Friedmann equation of pressureless matter leading to a decelerating universe with $a \propto t^{2/3}$ (Einstein-de Sitter solution). In the late universe, the term $\nu(\rho)$ dominates and causes the universe to accelerate. Freese and Lewis [29] justify the modified Friedmann equation (24) in relation to the existence of extra-dimensions. The usual Friedmann equation is modified as a consequence of embedding our universe as a three-dimensional surface (3-brane) in higher dimensions. Our approach provides another, simpler, justification of this equation from the ordinary four dimensional Einstein equations. Starting from the usual Friedmann equation (2) and considering a dark fluid at $T = 0$, or an adiabatic fluid, with the energy density given by $\epsilon = \rho c^2 + u(\rho)$ [see Eq. (5)], we obtain Eq. (24) with $\nu(\rho) = (8\pi G/3c^2)u(\rho)$. Therefore, the “new” term in the “modified” Friedmann equation (24) can be interpreted as the internal energy $u(\rho)$ of the dark fluid while the “ordinary” term $8\pi G\rho/3$ corresponds to its rest-mass energy density ρc^2 . Therefore, our approach provides a new justification of the Cardassian model. Inversely, the original justification of the Cardassian model, namely that the term $\nu(\rho)$ arising in the modified Friedmann equation (24) may result from the existence of extra-dimensions, could be a way to justify the logotropic model corresponding to $\nu(\rho) = -(8\pi G A/3c^2)[\ln(\rho/\rho_F) + 1]$. In this respect, we note that the logotropic model asymptotically yields an equation of state of the form $P \sim -\epsilon$ [see Eq. (9)]. Using the virial theorem, one can easily show that this equation of state arises from a long-range confining force $F_{ij} = -3U_0 r_{ij}^2$ that could be a fifth force or an effective description of higher dimensional physics [29].

³ The TF relation is sometimes justified by the MOND theory [28] which predicts a relation of the form $v_h^4 = G a_0 M_b$ between the asymptotic circular velocity and the baryon mass, where a_0 is a critical acceleration. Our results imply $a_0^{\text{th}} = 1.72 \times 10^{-10} \text{ m/s}^2$ which is close to the value $a_0^{\text{obs}} = (1.3 \pm 0.3) \times 10^{-10} \text{ m/s}^2$ obtained from the observations [10]. Using the results of the Appendix, we have $a_0^{\text{th}} \simeq H_0 c/4$ which explains why a_0 is of the order $H_0 c$. We emphasize, however, that we do not use the MOND theory in our approach.

B. Generalized thermodynamics

The logotropic equation of state was introduced phenomenologically in astrophysics by McLaughlin and Pudritz [30] to describe the internal structure and the average properties of molecular clouds and clumps. It was also studied by Chavanis and Sire [21] in the context of Tsallis generalized thermodynamics [31] where it was shown to correspond to a polytropic equation of state of the form $P = K\rho^\gamma$ with $\gamma \rightarrow 0$ and $K \rightarrow \infty$ in such a way that $A = \gamma K$ is finite. It is associated with a generalized entropy of the form

$$S_L = \int \ln \rho \, d\mathbf{r}, \quad (25)$$

which is called the Log-entropy [21]. The free energy can be written as $F_L = E - AS_L$, where $E = (1/2) \int \rho \Phi \, d\mathbf{r}$ is the gravitational energy. A critical point of F_L at fixed mass $M = \int \rho \, d\mathbf{r}$, determined by the Euler-Lagrange equation $\delta F_L - \mu \delta M = 0$, where μ is a Lagrange multiplier (chemical potential), leads to the Lorentzian-type distribution $\rho(\mathbf{r}) = 1/[\alpha + \Phi(\mathbf{r})/A]$, where $\alpha = -\mu/A$. We can check that this equation is equivalent to the equation of hydrostatic equilibrium with the logotropic equation of state (6). When combined with the Poisson equation, we recover the Lane-Emden equation (19). These considerations show that A can be interpreted as a generalized temperature. This is why we call it the logotropic temperature. As a result, the universality of A (which explains the constant values of Σ_0 , M_{300} and M_b/v_h^4) may be interpreted by saying that the universe is “isothermal”, except that isothermality does not refer to a linear equation of state but to a logotropic equation of state in a generalized thermodynamical framework. If our model is correct, it would be a nice confirmation of the interest of generalized thermodynamics [31] in physics and astrophysics.

C. Scalar field theory

The logotropic equation of state can also be justified from a scalar field theory. If we view the dark fluid as a scalar field representing BECs, its evolution is described, in the nonrelativistic regime appropriate to DM halos, by the Gross-Pitaevskii equation [32]. The GP equation associated with the logotropic equation of state takes the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi - Am \frac{1}{|\psi|^2} \psi, \quad (26)$$

where $\psi(\mathbf{r}, t)$ is the wave function. It is coupled to the Poisson equation $\Delta \Phi = 4\pi G |\psi|^2$. Eq. (26) can be viewed as a GP equation with an inverted quadratic potential, i.e. with the exponent -2 instead of $+2$ in the usual GP equation [33]. To our knowledge, this equation has not been introduced before. Using the

Madelung [34] transformation, this equation can be written in the form of fluid equations, called the quantum Euler equations [32], incorporating a quantum potential $Q = -(\hbar^2/2m)\Delta\sqrt{\rho}/\sqrt{\rho}$ due to the kinetic term $-(\hbar^2/2m)\Delta\psi$ (Heisenberg) and an isotropic pressure P due to the interaction term $Am|\psi|^{-2}\psi$. For the inverted quadratic potential, the pressure P is given by the logotropic equation of state (6). In the Thomas-Fermi approximation, one can neglect the quantum potential. In that case, we recover the classical Euler-Poisson equations. A steady state of these equations satisfies the condition of hydrostatic equilibrium. Furthermore, one can show that the free energy F_L and the mass M are conserved by the Euler-Poisson equations. As a result, a minimum of free energy F_L at fixed mass M is a stable steady state of the Euler-Poisson equations [32]. This makes a correspondance with generalized thermodynamics and shows the self-consistency of our approach. We also note that the GPP equations can be obtained as the nonrelativistic limit $c \rightarrow +\infty$ of the Klein-Gordon-Einstein equations [35]. The KG equation corresponding to the logotrope has a logarithmic potential $V(|\phi|) = -2A \ln |\phi|$ and writes

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2\Phi}{c^2}\right) \phi - \frac{2A}{|\phi|^2} \phi = 0. \quad (27)$$

V. CONCLUSION

We have proposed a heuristic unification of DM and DE in terms of a single dark fluid with a logotropic equation of state (LDF). According to our model, what we usually call DM corresponds to the rest-mass density of the dark fluid and what we usually call DE corresponds to the internal energy of the dark fluid.

At the cosmological scale, our model satisfies the same observational constraints as the Λ CDM model but avoids the cosmic coincidence problem (since DM and DE are the manifestation of a single dark fluid) and the cosmological constant problem (since there is no cosmological constant in our approach). It also has a nonzero speed of sound and a nonzero Jeans length (contrary to the Λ CDM model) which can explain the minimum size $R_{\min} \sim 10$ pc of DM halos.

At the galactic scale, the logotropic pressure balances gravitational attraction and solves the cusp problem and the missing satellite problem of the CDM model. On the other hand, the logotropic model generates a universal rotation curve that provides a good agreement with the Burkert profile up to the halo radius. Furthermore, it implies that the surface density of DM halos and the mass of dwarf halos are the same for all the halos, in agreement with the observations. It also implies the TF relation.

The most striking property of the logotropic model is the following. Using cosmological observations, we can obtain the value of the logotropic temperature $A =$

$2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}$ [see Eq. (12)]. It may be viewed as a fundamental constant since it actually depends on all the fundamental constants of physics \hbar , G , c , and Λ . Then, applying the logotropic model to DM halos, and using this value of A , we can obtain the value of Σ_0 [see Eq. (21)], M_b/v_h^4 [see Eq. (22)] and M_{300} [see Eq. (23)] which are in perfect agreement with the observations. Therefore, the logotropic model is able to account both for cosmological and galactic observations remarkably well. This may be a hint that DM and DE are the manifestation of a unique dark fluid.

Finally, we have sketched some possible justifications of the logotropic equation of state in relation to the existence of extra-dimensions (Cardassian model), in relation to Tsallis generalized thermodynamics, and in relation to scalar field theory and BECs.

The fact that the Planck density ρ_P enters in the logotropic equation of state (6) designed to model DM and DE is intriguing. It suggests that quantum mechanics manifests itself at the cosmological scale in relation to DE. This may be a hint for a fundamental theory of quantum gravity. This also suggests that the logotropic equation of state may be the limit of a more general equation of state providing a possible unification of DE (ρ_Λ) in the late universe and inflation (vacuum energy ρ_P) in the primordial universe. These open questions are a

strong incentive to study the logotropic model further in future works. The phantom properties of the logotropic model will be discussed in a specific paper.

Appendix A: Expression of the observational quantities in terms of the fundamental constants

We enlight the remarkable feature that, in our theory, all the observational quantities can be predicted in terms of fundamental constants such as \hbar , G , c , H_0 , and $\Omega_{m,0}$. We have $B = 1/[\ln(8\pi c^5/3\Omega_{m,0}\hbar GH_0^2) - 1]$. We introduce the notation $\chi = [3B(1 - \Omega_{m,0})/2]^{1/2} = 6.20 \times 10^{-2}$. Then $A = \chi^2 c^2 H_0^2 / 4\pi G = 3.06 \times 10^{-4} c^2 H_0^2 / G$, $\Sigma_0 = \chi \xi_h H_0 c / 4\pi G = 2.89 \times 10^{-2} H_0 c / G$, $M_h/r_h^2 = \chi \theta'(\xi_h) H_0 c / G = 4.30 \times 10^{-2} H_0 c / G$, $v_h^2/r_h = \chi \theta'(\xi_h) H_0 c = 4.30 \times 10^{-2} H_0 c$, $v_h^4/M_b = \chi \theta'(\xi_h) G H_0 c / f_b = 4.30 \times 10^{-2} G H_0 c / f_b$, $a_0 = \chi \theta'(\xi_h) H_0 c / f_b = 4.30 \times 10^{-2} H_0 c / f_b$, and $M_{300}/r_u^2 = \chi H_0 c / \sqrt{2} G = 4.39 \times 10^{-2} H_0 c / G$. Noting that $\epsilon_\Lambda = \Lambda c^2 / 8\pi G$, $\epsilon_\Lambda = (1 - \Omega_{m,0})\epsilon_0$ and $H_0^2 = (8\pi G/3c^2)\epsilon_0$, we obtain $\Lambda = 3(1 - \Omega_{m,0})H_0^2 = 2\chi^2 H_0^2 / B = 1.13 \times 10^{-35} \text{ s}^{-2}$. Therefore, the observational quantities can be expressed equivalently in terms of \hbar , G , c , Λ and $\Omega_{m,0}$.

-
- [1] F. Zwicky, *Helv. Phys. Acta* **6**, 110 (1933)
 - [2] V.C. Rubin, W.K. Ford, N. Thonnard, *Astrophys. J.* **238**, 471 (1980); M. Persic, P. Salucci, F. Stel, *Mon. Not. R. astr. Soc.* **281**, 27 (1996)
 - [3] A.G. Riess et al., *Astron. J.* **116**, 1009 (1998); S. Perlmutter et al., *ApJ* **517**, 565 (1999); P. de Bernardis et al., *Nature* **404**, 995 (2000); S. Hanany et al., *ApJ* **545**, L5 (2000)
 - [4] A. Einstein, *Sitz. König. Preu. Akad. Wiss.* **1**, 142 (1917)
 - [5] J.F. Navarro, C.S. Frenk, S.D.M. White, *Mon. Not. R. astr. Soc.* **462**, 563 (1996)
 - [6] A. Burkert, *Astrophys. J.* **447**, L25 (1995)
 - [7] G. Kauffmann, S.D.M. White, B. Guiderdoni, *Mon. Not. R. astr. Soc.* **264**, 201 (1993); A. Klypin, A.V. Kravtsov, O. Valenzuela, *Astrophys. J.* **522**, 82 (1999); M. Kamionkowski, A.R. Liddle, *Phys. Rev. Lett.* **84**, 4525 (2000)
 - [8] F. Donato et al., *Mon. Not. R. Astron. Soc.* **397**, 1169 (2009)
 - [9] R.B. Tully, J.R. Fisher, *Astron. Astrophys.* **54**, 661 (1977)
 - [10] S.S. McGaugh, *Astron. J.* **143**, 40 (2012)
 - [11] L.E. Strigari et al., *Nature* **454**, 1096 (2008)
 - [12] H.J. de Vega, P. Salucci, N.G. Sanchez, *Mon. Not. R. Astron. Soc.* **442**, 2717 (2014)
 - [13] P.H. Chavanis, M. Lemou, F. Méhats, *Phys. Rev. D* **91**, 063531 (2015); P.H. Chavanis, M. Lemou, F. Méhats, [arXiv:1409.7840]
 - [14] A. Suárez, V.H. Robles, T. Matos, *Astrophys. Space Sci. Proc.* **38**, 107 (2014); T. Rindler-Daller, P.R. Shapiro, *Astrophys. Space Sci. Proc.* **38**, 163 (2014); P.H. Chavanis, *Self-gravitating Bose-Einstein condensates*, in *Quantum Aspects of Black Holes*, edited by X. Calmet (Springer, 2015)
 - [15] P. Steinhardt, in *Critical Problems in Physics*, edited by V.L. Fitch and D.R. Marlow (Princeton University Press, Princeton, NJ, 1997)
 - [16] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989); T. Padmanabhan, *Phys. Rep.* **380**, 235 (2003)
 - [17] G. Lemaitre, *Proc. Nat. Acad. Sci.* **20**, 12 (1934); A.D. Sakharov, *Dokl. Akad. Nauk SSSR* **177**, 70 (1967); Ya. B. Zeldovich, *Sov. Phys. Uspek.* **11**, 381 (1968)
 - [18] B. Ratra, J. Peebles, *Phys. Rev. D* **37**, 321 (1988)
 - [19] A. Kamenshchik, U. Moschella, V. Pasquier, *Phys. Lett. B* **511**, 265 (2001)
 - [20] H.B. Sandvik, M. Tegmark, M. Zaldarriaga, I. Waga, *Phys. Rev. D* **69**, 123524 (2004); Z.H. Zhu, *Astron. Astrophys.* **423**, 421 (2004)
 - [21] P.H. Chavanis, C. Sire, *Physica A* **375**, 140 (2007)
 - [22] S. Weinberg, *Gravitation and Cosmology* (John Wiley, 2002)
 - [23] P.H. Chavanis, *Eur. Phys. J. Plus* **129**, 222 (2014)
 - [24] R.R. Caldwell, *Phys. Lett. B* **545**, 23 (2002)
 - [25] C. Destri, H.J. de Vega, N.G. Sanchez, *New Astronomy* **22**, 39 (2013)
 - [26] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover, 1958)
 - [27] A. Burkert, arXiv:1501.06604 (2015)
 - [28] M. Milgrom, *Astrophys. J.* **270**, 365 (1983)
 - [29] K. Freese, M. Lewis, *Phys. Lett. B* **540**, 1 (2002)
 - [30] D. McLaughlin, R. Pudritz, *Astrophys. J.* **469**, 194 (1996)

- [31] C. Tsallis, Introduction to Nonextensive Statistical Mechanics (Springer, 2009)
- [32] P.H. Chavanis, Phys. Rev. D **84**, 043531 (2011)
- [33] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Rev. Mod. Phys. **71**, 463 (1999)
- [34] E. Madelung, Zeit. F. Phys. **40**, 322 (1927)
- [35] A. Suárez, P.H. Chavanis, Phys. Rev. D **92**, 023510 (2015)